IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Matrix and tensor constructions from a generic SU(n) vector

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1972 J. Phys. A: Gen. Phys. 5 1043

(http://iopscience.iop.org/0022-3689/5/7/014)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.73
The article was downloaded on 02/06/2010 at 04:38

Please note that terms and conditions apply.

J. Phys. A: Gen. Phys., Vol. 5, July 1972. Printed in Great Britain. © 1972.

Matrix and tensor constructions from a generic SU(n) vector

K J BARNES† and R DELBOURGO‡

† Department of Physics, Queen Mary College, Mile End Road, London E1 4NS, UK
‡ Department of Physics, Imperial College, London SW7 2BZ, UK

MS received 10 February 1972

Abstract. Projection operators appropriate to the general multispinor representations of SU(n) are constructed in a systematic way from the components of a single generic SU(n) vector transforming as the adjoint representation. The techniques have been devised with the problems of writing explicit forms for finite SU(n) rotations and nonlinear chiral Lagrangians kept specifically in mind. In particular, the general second rank tensors constructed from a single vector are found, counted, and exhibited in a very tractable form.

1. Introduction

Recently new attempts have been made to understand the nonsymmetric pieces of the hadronic Hamiltonian in terms of spontaneous breakdown of the basic chiral

$$K(3) = SU(3)\otimes SU(3)$$

symmetry (Dashen 1971, Nuyts 1971). These ideas lead, via a minimization procedure, to the requirement of constructing explicitly the transformations of low dimensional SU(3) representations through a *finite* angle specified by the components of a single octet vector. It has long been realized that the elusive solution to the problem of obtaining explicit *closed* forms for nonlinear realizations, and hence of forming chiral invariant nonlinear Lagrangians, might also follow from the same techniques (Macfarlane *et al* 1970). For this latter problem (Gasiorowicz and Geffen 1969) it has recently been emphasized (Barnes 1972) that it is also necessary to construct general second rank tensors from the given octet, and that one must be able to form inverses and products of such tensors. These two separate developments in the theory of hadronic interactions have shown the need for handling the same piece of mathematical machinery, and SU(3) treatments of these problems have been given of late (Rosen 1971, Barnes *et al* 1972). In all versions the construction of matrices which act as projection operators proves to be a key concept, and much reliance is placed upon the ideas of charge and special vectors introduced by Michel and Radicati (1968) beyond the SU(2) level.

In this paper we show how to extend these ideas to treat all problems of this general type in a systematic and coherent way at the SU(n) level. The notation and basic concepts required for our description are introduced in § 2. In § 3 we show how to find the basic projectors for any representation when only a single generic vector transforming as the adjoint representation is available, and we obtain two convenient base sets of $(n-1)$ orthonormal vectors from the given one. Section 4 contains a solution to the problem of constructing second rank tensors from the vectors, and a particularly useful example is presented in § 5 to illustrate the utility of our results.

2. Notation

We adopt the language and notation most familiar to high energy physicists. A detailed review may be found in the article by Matthews (1967), and we now list the essential results for our purposes. Given the Lie algebra of SU(n), we may take as a basis a set of (n^2-1) traceless hermitian $n \times n$ matrices λ_i with the product law

$$\lambda_i\lambda_j = (d_{ijk}+\mathrm{i}f_{ijk})\lambda_k+2\delta_{ij}/n \tag{2.1}$$

and where the equations

$$(\lambda_i)_{KL}(\lambda_i)_{MN} = 2(\delta_{KN}\delta_{ML}-\delta_{MN}\delta_{KL}/n) \tag{2.2}$$

express the completeness property. Here the indices $K, L, M, \ldots$ range from 1 to n, while the indices $i, j, k, \ldots$ run from 1 to n^2-1. Repeated indices are summed except where indicated in the text. All indices are taken as subscripts and we reserve superscripts to indicate powers. Note that $i, j, k, \ldots$ refer to the group SU(n)/Z(n), where Z(n) is the centre of SU(n), rather than SU(n) itself. Moreover, the (n^2-1) components obtained by contracting the indices of the λ matrices against a double indexed mixed spinor may be regarded as cartesian components of a vector in (n^2-1) real dimensions, since SU(n)/Z(n) is isomorphic to a subgroup of real rotations in (n^2-1) dimensions leaving $(n-1)$ polynomial forms invariant.

Clearly, from equation (2.1), the matricies $\frac{1}{2}\lambda_i$ represent the generators of SU(n) in the fundamental representation and f_{ijk} are the structure constants. Furthermore, it follows from (2.1) that the real quantities f_{ijk} and d_{ijk} are respectively totally antisymmetric and symmetric. It will also prove convenient to introduce the alternative set of n^2 matrices

$$(F_{AB})_{MN} = \delta_{MA}\delta_{NB} \tag{2.3}$$

which obey the product law

$$F_{AB}F_{CD} = \delta_{BC}F_{AD} \tag{2.4}$$

and thus generate the Lie algebra of U(n) on commutation. Since commutators are unaffected by tracing, the n^2-1 traceless matrices in this set generate SU(n), and the precise relationships are

$$(F_{AB})_{MN} = F_{AB\,i}(\lambda_i)_{MN} \tag{2.5}$$

or

$$F_{AB\,i} = \tfrac{1}{2}\,\mathrm{Tr}(F_{AB}\lambda_i). \tag{2.6}$$

By the Cayley–Hamilton theorem (Birkhoff and Maclane 1965) every hermitian $n \times n$ matrix M satisfies its own characteristic equation

$$\prod_{A=1}^{n}(M-m_A) = 0 \tag{2.7}$$

where the m_A are the eigenvalues. This equation may be expanded in the form

$$M^n+\sum_{A=1}^{n}c_AM^{n-A} = 0 \tag{2.8}$$

where the c_A are real symmetric functions of the eigenvalues and obey inequalities (discriminant conditions) due to the reality of the m_A. When the matrix is traceless, the coefficient c_1 vanishes, and we restrict ourselves to this case henceforth. Such a matrix has the expansion

$$M_{KL} = m_i(\lambda_i)_{KL} \tag{2.9}$$

giving us the n^2-1 components m_i of a vector transforming as the adjoint representation. In the terminology used by Michel and Radicati (1968), the vector is said to be generic (or belong to the generic stratum) if all eigenvalues are distinct. For the generic case the minimal polynomial for the matrix is the characteristic equation itself, so that the n vectors with components

$$M_{Ai} \equiv \tfrac{1}{2}\operatorname{Tr}(M^A\lambda_i) \tag{2.10}$$

given in terms of the powers of the matrix are a linearly independent set. Moreover with such a matrix, the c_A can be taken as $(n-1)$ independent invariants and are related to the alternative choice (MacMahon 1960)

$$s_A = \operatorname{Tr}(M^A) = \sum_{B=1}^{n} (m_B)^A \equiv \sum_{B=1}^{n} m_{AB} \tag{2.11}$$

by simple determinants. Observe that $m_i = m_{1i}$, and $s_1 = 0$, so that the $(n-1)$ objects are easy to identify in each case.

It is perhaps worth emphasizing at this point that if one chooses to start some calculation by specifying the components m_i then in order to apply the results which are presented in this paper it is necessary to find the eigenvalues as a first step. In general this involves solving an algebraic characteristic equation of degree n, and we therefore recommend an alternative starting point which involves specification of the eigenvalues. Of course, for the cases of SU(2) and SU(3) so much literature already exists that our advice is not appropriate. Fortunately the required connecting relationships between the two descriptions are not too hard to find for these cases, and have already been stated in closed form (Barnes 1972, Barnes *et al* 1972).

3. The basic projection operators

The crucial step, which is a common feature of this and all previous approaches (Rosen 1971, Barnes *et al* 1972), is the resolution of the powers of the matrix M in the form

$$M^A = (m_B)^A P_B \equiv m_{AB} P_B \tag{3.1}$$

where the P_B are n hermitian matrices, each $n \times n$, with the properties

$$P_A P_B = \delta_{AB} P_B \qquad \text{(no sum)} \tag{3.2}$$

$$\operatorname{Tr}(P_A) = 1 \tag{3.3}$$

and

$$\sum_{A=1}^{n} P_A = 1 \tag{3.4}$$

where 1 is the unit $n \times n$ matrix. It is, of course, the fact that the P_A are projectors that makes all subsequent calculations tractable. Notice that the inverse of equation (3.1) may be written in the form

$$P_A = m_{AB}^{-1} M^B. \tag{3.5}$$

This particular formula is especially useful in making contact between the present fairly formal manipulations and the explicit forms occurring in practice when n assumes low values. It should be stressed that the procedure does not, of course, diagonalize the matrices.

Now that we have the n projectors of the fundamental representation it is a simple matter to construct the projectors of higher representations by forming the direct symmetrized products of the fundamental ones appropriate to the corresponding multi-spinor representations. For example, let us consider the projection operators which act on the components of the $n(n+1)(n+2)/6$ dimensional irreducible multispinor having three symmetric indices (the decuplet at the SU(3) level). We define

$$P_{(ABC)} = \tfrac{1}{6} \sum P_A \otimes P_B \otimes P_C \tag{3.6}$$

where the summation is over all permutations of ABC to ensure symmetry and the factor $\frac{1}{6}$ is for purposes of normalization. In (3.6) we have omitted the multispinor indices on which these projectors operate for clarity. From equations (3.2) and (3.3) we get at once

$$\begin{aligned} P_{(ABC)} P_{(DEF)} &= \tfrac{1}{36} \sum \sum P_A P_D \otimes P_B P_E \otimes P_C P_F \\ &= \delta_{(ABC)(DEF)} P_{(DEF)} \qquad \text{(no sum)} \end{aligned} \tag{3.7}$$

and

$$\mathrm{Tr}(P_{(ABC)}) = \mathrm{Tr}(P_A)\,\mathrm{Tr}(P_B)\,\mathrm{Tr}(P_C) = 1 \tag{3.8}$$

where the sets (ABC) and (DEF) in equation (3.7) act as single indices in the final expression. It should be clear from our earlier remarks, and from this example, how to build up projectors for all representations of SU(n).

As mentioned before it is often convenient to work directly with vector and tensor representations of SU(n)/Z(n) rather than with multispinors. With this in mind, we consider the equations

$$(P_A)_{MN} = P_{Ai}(\lambda_i)_{MN} + \delta_{MN}/n \tag{3.9}$$

or

$$P_{Ai} = \tfrac{1}{2}\,\mathrm{Tr}(P_A \lambda_i) \tag{3.10}$$

which define the components of the P_A in the cartesian basis. Substituting (3.10) into (3.2) and (3.4) we find

$$f_{ijk} P_{Aj} P_{Bk} = 0 \tag{3.11}$$

$$P_{Aj} P_{Bj} = \tfrac{1}{2}(\delta_{AB} - 1/n) \tag{3.12}$$

$$d_{ijk} P_{Aj} P_{Bk} = \delta_{AB} P_{Bi} - (P_{Ai} + P_{Bi})/n \qquad \text{(no sum on } B) \tag{3.13}$$

and

$$\sum_{A=1}^{n} P_{Ai} = 0 \tag{3.14}$$

as the basic properties of the vector projectors. Notice that although these vectors lie in a charge stratum (ie the minimal polynomial of their matrices is of degree two) they are not the charge vectors defined by Michel and Radicati (1968). To arrive at the connection with the work of these authors, we observe that

$$(P_{Ai}-P_{Bi})(P_{Ai}-P_{Bi}) = 1 \qquad \text{(no sum on } A \text{ and } B) \tag{3.15}$$

$$d_{ijk}(P_{Aj}-P_{Bj})(P_{Ak}-P_{Bk}) = P_{Ai}+P_{Bi} \qquad \text{(no sum on } A \text{ and } B) \tag{3.16}$$

$$(P_{Ai}+P_{Bi})(P_{Ai}+P_{Bi}) = 1-2/n \qquad \text{(no sum on } A \text{ and } B) \tag{3.17}$$

$$d_{ijk}(P_{Aj}+P_{Bj})(P_{Ak}+P_{Bk}) = (P_{Ai}+P_{Bi})(1-4/n) \qquad \text{(no sum on } A \text{ and } B) \tag{3.18}$$

so that the differences of the P_{Ai} are the special vectors of Michel and Radicati and the sums of the P_{Ai} are their associated charge vectors up to a normalizing factor. These distinctions are of no consequence until $n \geqslant 4$ and, in particular, for the important case of $n = 3$, we see that the P_{Ai} and the charge vectors are identical apart from a trivial numerical factor. For larger values of n the charge vectors do not directly (ie separately) define irreducible subspaces, nor do they have the same little group as the P_{Ai}. However, the charge vectors and the P_{Ai} are always linearly related as exhibited above, so that from an algebraic viewpoint they are equally useful.

From equations (3.12) and (3.14) it is clear that we are dealing with a set of n linearly dependent vectors treated in a symmetrical manner. However, for many purposes it is more convenient to use $(n-1)$ orthonormal vectors as a base set. We now introduce two such sets, each of which has its own particular advantages. One set is defined by

$$p_{Ai} = 2^{1/2}\{P_{Ai}-(1+n^{1/2})^{-1}P_{ni}\} \tag{3.19}$$

and the other set by

$$\begin{aligned} p'_{Ai} &= \left(\frac{2}{A(A+1)}\right)^{1/2} \sum_{B=1}^{A} (P_{Bi}-P_{A+1\,i}) \\ &= \left(\frac{2}{A(A+1)}\right)^{1/2} \left(\sum_{B=1}^{A} P_{Bi}-AP_{A+1\,i}\right) \end{aligned} \tag{3.20}$$

where for the p_{Ai} or p'_{Ai} the label A ranges from 1 to $n-1$. The first set were found by inspection, and the second set follow from the Schmidt orthogonalization procedure (Courant and Hilbert 1963). It is precisely the latter means of construction which make the p'_{Ai} especially useful; thus successive vectors are made orthogonal to previous ones without reference to subsequent ones, and therefore the first $r < n$ can be taken at once as a basis for the corresponding problem at the SU$(r+1)$ level. In particular, the matrices

$$(p'_A)_{MN} = p'_{Ai}(\lambda_i)_{MN} \tag{3.21}$$

are traceless, and in the diagonal basis where

$$(P_A)_{MN} = \delta_{AM}\delta_{AN} \qquad \text{(no sum on } A) \tag{3.22}$$

are the familiar canonical choice in much of the literature (Matthews 1967). Inspection of equation (3.19) instead, shows an explicit dependence upon n, so that the p_{Ai} have no equivalent property. Nevertheless the p_{Ai} have the advantage of being more simply related to the P_{Ai}, and this allows for a particularly neat treatment of nonlinear chiral Lagrangians (Barnes *et al* 1971) where use of the P_{Ai} is crucial.

The basic machinery appropriate to any multispinor or tensor representation has now been set up, and we have discovered two convenient bases for the vectors of the adjoint representation. The remaining problem of immediate physical interest is to count the number of independent second rank tensors we can construct from a single generic vector, and to build the general tensor of this type in a manner which makes all subsequent manipulations (inversion, square rooting, etc) possible and trivial. This is the subject of the next section.

4. The second rank tensors

When specifying the nonlinear realizations of chiral algebras (Coleman *et al* 1969, Isham 1969) and subsequently constructing hadronic Lagrangians (Callan *et al* 1969), the second rank tensors play a vital role (Weinberg 1968). This has been discussed extensively in the literature (Macfarlane *et al* 1970, Barnes 1972), and we content ourselves here with a statement of what is required. We need to build general second rank tensors from a generic hermitian vector of $SU(n)/Z(n)$ and be able to form inverses and simple products of such tensors viewed as $(n^2-1)\times(n^2-1)$ matrices. This last requirement strongly suggests that we include in the base set of such tensors the (n^2-1) which behave as projectors.

As a first step let us consider the $n(n-1)+1$ matrices each $(n^2-1)\times(n^2-1)$, defined by

$$(P_{AB})_{ij} \equiv P_{AiBj} \equiv \tfrac{1}{2}\operatorname{Tr}(P_A\lambda_iP_B\lambda_j) \qquad A\neq B \tag{4.1}$$

and

$$I_{ij} = \tfrac{1}{2}\operatorname{Tr}(P_A\lambda_iP_A\lambda_j) \tag{4.2}$$

where the P_A are the basic projection operators defined by our generic vector. In order to exhibit the properties of these matrices we apply the two identities

$$\tfrac{1}{2}\operatorname{Tr}(\lambda_iX)\operatorname{Tr}(\lambda_iY) = \operatorname{Tr}(XY)-\operatorname{Tr}X\operatorname{Tr}Y/n \tag{4.3}$$

and

$$\tfrac{1}{2}\operatorname{Tr}(\lambda_iX\lambda_iY) = \operatorname{Tr}X\operatorname{Tr}Y-\operatorname{Tr}(XY)/n \tag{4.4}$$

which follow immediately from the completeness relation (2.2). X and Y are arbitrary $n\times n$ matrices. One learns that

$$II = I \tag{4.5}$$

$$IP_{AB} = 0 = P_{AB}I \tag{4.6}$$

and

$$P_{AB}P_{CD} = \delta_{AC}\delta_{BD}P_{AB} \qquad \text{(no sum)} \tag{4.7}$$

where matrix indices are omitted and a matrix product is implied on the left hand side of the equations. Henceforce we will omit such indices without comment whenever a gain in clarity can be achieved without causing confusion or ambiguity. From equation (4.2) we obtain further

$$\operatorname{Tr}I = n-1 \tag{4.8}$$

$$\operatorname{Tr}P_{AB} = 1. \tag{4.9}$$

Also we can derive the completeness relation

$$\sum_{A\neq B} P_{AB}+I = 1 \tag{4.10}$$

where 1 denotes the unit matrix with components δ_{ij}, by using the product rule (2.1) and equation (3.4). Thus the matrices (4.1) and (4.2) have the properties of projectors, and I can be reduced further into a sum of $(n-1)$ matrices which act as projection operators, as we shall see.

Now we are in a position to construct the most general second rank tensor from our generic vector. We start by resolving the vector into components

$$m_i = \tfrac{1}{2}\operatorname{Tr}(\lambda_i M) = \tfrac{1}{2} m_A \operatorname{Tr}(\lambda_i P_A) \tag{4.11}$$

and similarly all powers of M as in equation (3.1). The problem then reduces to constructing the most general second rank tensor T_{ij} from the invariants m_A and the matrices P_A, where the free tensor labels are carried by the λ matrices. Evidently T_{ij} may be written as a sum (with coefficients as functions of the m_A) over all second rank tensors constructed similarly from the P_A and λ matrices alone. These last tensors all take the form of sums of products of traces over the $n\times n$ matrices taken as products of P_A and λ matrices. As an aid to visualization consider the typical example

$$\operatorname{Tr}(\lambda_i P_A P_B \lambda_l \lambda_k P_D \lambda_l)\operatorname{Tr}(\lambda_j P_E \lambda_k P_C) \tag{4.12}$$

where the only free *tensor* indices are i and j, all others being summed. The projector labels A, B, C, D, E are also free. The completeness properties (4.3) and (4.4) make it possible to simplify expressions like (4.12) by rewriting them as sums of similar ones involving no λ matrices other than λ_i and λ_j; thus the repeated index k may be removed by using (4.3), while the sum over l is eliminated by using (4.4); next we may reduce any two contiguous P_A by their basic projection property (3.2).

Only two basic types of tensor survive this analysis

$$\tfrac{1}{4}\operatorname{Tr}(\lambda_i P_A)\operatorname{Tr}(\lambda_j P_B) = P_{Ai}P_{Bj} \tag{4.13}$$

and

$$\tfrac{1}{2}\operatorname{Tr}(P_A \lambda_i P_B \lambda_j). \tag{4.14}$$

Once again the completeness property of the λ matrices plays a crucial role, for we learn the multiplication rules

$$I_{ij}P_{Aj} = P_{Ai} = P_{Aj}I_{ji} \tag{4.15}$$

and

$$P_{BiCj}P_{Aj} = 0 = P_{Aj}P_{BjCi}. \tag{4.16}$$

Therefore the $P_{Ai}P_{Bj}$ lie in the subspace of $(n^2-1)\times(n^2-1)$ matrices projected out on multiplication by I from both left and right. Also, because the trace of I was found to be $(n-1)$, the $(n-1)^2$ independent matrices with components

$$(p_{AB})_{ij} \equiv p_{Ai}p_{Bj} \tag{4.17}$$

span this subspace. Moreover, since the vectors p_{Ai} are orthonormal, the multiplication law for such matrices is

$$p_{AB}p_{CD} = \delta_{BC}p_{AD} \tag{4.18}$$

and comparison with equation (2.4) reveals that we are now dealing with the familiar rule which defines the matrices generating U$(n-1)$. (We emphasize, however, that the p_{AB} are $(n^2-1)\times(n^2-1)$ matrices.) In particular, the remaining $(n-1)$ matrices which act as projectors into which I may be decomposed are to be found among this set, and to specify them would require both a generic $\mathrm{SU}(n-1)$ vector and the techniques we have developed here.

We thus have a basis for all tensors in equation (4.13) so let us turn our attention to the remaining ones (4.14). Unless $B = A$, the tensor in (4.14) is identically the projector P_{AiBj} of (4.1), so it only remains to examine the tensors

$$(I_A)_{ij} = \tfrac{1}{2}\mathrm{Tr}(P_A\lambda_i P_A\lambda_j) \qquad \text{(no sum)}. \tag{4.19}$$

But on applying (4.3) yet again we see that

$$II_A = I_A I \tag{4.20}$$

and

$$P_{AB}I_C = 0 = I_C P_{AB}. \tag{4.21}$$

Hence I_C can be expressed in terms of the p_{AB}. In fact, since the I_C involve just P_C, we deduce that

$$(I_A)_{ij} = 2P_{Ai}P_{Aj} \qquad \text{(no sum)} \tag{4.22}$$

a conclusion which is obvious in a basis which diagonalizes the P_A.

The analysis is now complete. A general second rank tensor (matrix) may be expanded in the form

$$T_{ij} = T_{AB}(P_{AB})_{ij} + t_{AB}(p_{AB})_{ij} \tag{4.23}$$

where P_{AB} and p_{AB} are the $(n-1)(2n-1)$ independent tensors defined in (4.1) and (4.17). The expansion coefficients T_{AB} and t_{AB} depend solely on the invariants m_A of the generic vector. Furthermore, the basic tensors P_{AB} and p_{AB} have such simple product rules that subsequent manipulations of T_{ij} become quite tractable if not trivial.

At this point, it is worth remarking that one could extend these methods for counting and constructing higher rank cartesian tensors. For instance, a third rank tensor T_{ijk} may be expanded at most in terms of the set

$$\mathrm{Tr}(\lambda_i P_A \lambda_j P_B \lambda_k P_C) \qquad A \neq B \neq C \tag{4.24}$$

$$p_{Ak}\,\mathrm{Tr}(P_B\lambda_i P_C \lambda_j) \qquad B \neq C \tag{4.25}$$

$$p_{Ak}\,\mathrm{Tr}(P_A\lambda_i P_B \lambda_j) \qquad B \neq A \tag{4.26}$$

$$p_{Ai}p_{Bj}p_{Ck} \qquad \text{and perms} \tag{4.27}$$

a total of $(n-1)(6n^2-9n+1)$ tensors which are expected to be linearly independent.

5. An example and conclusions

We have shown how to write projection operators, defined in terms of a single generic $\mathrm{SU}(n)$ vector (specified by its eigenvalues), appropriate to arbitrary rank multispinor representations. Also, we have shown how to work directly in the $\mathrm{SU}(n)/\mathrm{Z}(n)$ framework and have constructed the most general second rank tensor. As an illustrative example to show the usefulness and power of our methods, we shall conclude this paper by deriving two simple expressions that are of importance in both types of calculation mentioned in the introduction. We shall exhibit how these expressions lead simply to a well known result, thus giving a check on our techniques. From the generic expansion (4.11) we may write

$$\mathrm{i}f_{ikj}m_k = \tfrac{1}{4}m_A\,\mathrm{Tr}(\lambda_j\lambda_i P_A - \lambda_i\lambda_j P_A) \tag{5.1}$$

using the multiplication rule (2.1). Inserting the unit operator as in equation (3.4) between contiguous matrices, we obtain

$$(F_m)_{ij} = \mathrm{i}f_{ikj}m_k = \sum_A\sum_B \tfrac{1}{2}(m_B - m_A)(P_{AB})_{ij} \tag{5.2}$$

as our first expression. Next, consider the unitary transformations of a basic spinor N_A specified by

$$N_A \to U_{AB}N_B \tag{5.3}$$

with

$$U = \exp(\tfrac{1}{2}\mathrm{i}\theta_j\lambda_j) = P_A\exp(\tfrac{1}{2}\mathrm{i}\theta_A) \tag{5.4}$$

where the last step is the equivalent of equation (4.11) and where the real components of the angles θ must satisfy the condition of unimodularity

$$\sum_A \theta_A = 0. \tag{5.5}$$

It is easy to prove that the transformation induces a corresponding orthogonal one (Macfarlane *et al* 1970)

$$v_i \to R_{ij}v_j \tag{5.6}$$

where v_i are the components of any vector in the adjoint representation and

$$R_{ij} = \tfrac{1}{2}\mathrm{Tr}(U^{-1}\lambda_i U\lambda_j). \tag{5.7}$$

In our present notation this last expression may be immediately written in the required form

$$R = I + \sum_A\sum_B P_{AB}\exp\{\tfrac{1}{2}\mathrm{i}(\theta_B - \theta_A)\} \tag{5.8}$$

which can be compared and contrasted with the results obtained at the SU(3) level (Barnes 1972, Rosen 1971). Of course, if we define

$$(F_\theta)_{ij} = \mathrm{i}f_{ikj}\theta_k \tag{5.9}$$

then the familiar result

$$R = \exp(\mathrm{i}F_\theta) \tag{5.10}$$

follows at once from (5.5) and provides a useful check on our work.

As mentioned in the introduction, the immediate applications of these techniques in high energy physics are to the construction of chiral Lagrangians (Gasiorowicz and Geffen 1969, Callan *et al* 1969) and to the minimization problems of the Dashen type (Dashen 1971).

The results embodied in (5.2), (5.4) and (5.8) provide a framework for studying problems of the latter kind. Dashen showed that the application of a generalized minimum energy principle to the vacuum expectation value of the Hamiltonian in chiral theories placed restrictions on the ranges of the parameters specifying the symmetry breaking. Without such restrictions completely unphysical results (such as negative values for squares of masses) are predicted. To find the allowed class of Hamiltonians it is necessary to write explicit closed forms specifying rotations of simple representations of the chiral algebra. The expressions we have indicated above certainly give such forms for the most important cases, and in particular for the $(3,3^*)+(3^*,3)$ and $(1,8)+(8,1)$ representations of K(3), and have the added advantage that they are easy to manipulate. Moreover, the expressions for the finite rotations show a smooth transition from the generic case to rotations specified by vectors on more singular strata. This latter point may well be of importance, for Michel and Radicati (1968) give strong reasons for expecting physically significant minima on nongeneric strata. A particular scheme of this type is presently under consideration, and we hope to present the main physical consequences at length in the near future.

If θ is generalized (Barnes *et al* 1971) to become a function of the invariants $m_{\{}$, then (5.4) gives an elegant solution to the problem of constructing nonlinear chiral invariant Lagrangians. The basic requirement has long been known (Coleman *et al* 1969) to be the construction of the most general $(n \times n)$ unitary unimodular matrix from a single hermitian adjoint vector, and (5.4) is then precisely what is needed. Once this is available, the general theory may be applied more or less directly to construct the Lagrangians, and the material we presented in § 4 is the appropriate machinery for handling the expressions which arise. In particular, the covariant derivatives (of fields other than the M^i) which play a crucial role in this theory can be expressed directly in terms of a rational form in R and the unit matrix. This is treated at length in Barnes *et al* (1971) to which we refer the reader for further details. In point of fact, the construction (Sarkar 1971, pp 95–140) of the chiral K(3) nonlinear Lagrangians preceded, and led directly to, the present work. However, the present ideas have made possible a far more coherent presentation of those results (Barnes *et al* 1971), and allow for the extension to the cases of K(4) and K(6) suggested by spin symmetries (Matthews 1967, Delbourgo 1968, Delbourgo *et al* 1969).

Acknowledgments

One of the authors (KJB) is grateful to the SRC for supporting a meeting on the Dynamical Stability of Symmetry Breaking at Queen Mary College. Ideas presented at that meeting had a marked influence on the development of the present work.

References

Barnes K J 1972 *J. Phys.* A: *Gen. Phys.* **5** 830–7
Barnes K J, Dondi P H and Sarkar S C 1972 *J. Phys.* A: *Gen. Phys.* **5** 555–62
—— 1971 *Queen Mary College Preprint*
Birkhoff G and Maclane S 1965 *A Survey of Modern Algebra* (New York: Macmillan) p 299
Callan C G, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247–50
Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2239–47
Courant R and Hilbert D 1963 *Methods of Mathematical Physics* vol 1 (New York: Interscience) p 4
Dashen R 1971 *Phys. Rev.* D **3** 1879–90
Delbourgo R 1968 *J. math. Phys.* **9** 1936–9
Delbourgo R, Koller K and Williams R 1969 *J. math. Phys.* **10** 957–63
Gasiorowicz S and Geffen D A 1969 *Rev. mod. Phys.* **41** 531–74
Isham C J 1969 *Nuovo Cim.* A **59** 356–76
Macfarlane A J, Sudbery A and Weisz P 1970 *Proc. R. Soc.* A **314** 217–50

MacMahon P A 1960 *Combinatory Analysis* (New York: Chelsea) chap 1
Matthews P T 1967 *High Energy Physics* vol 1 (New York: Academic Press) pp 392–483
Michel L and Radicati L A 1968 *Symmetry Principles at High Energy* (New York: Benjamin) pp 19–48
Nuyts J 1971 *Phys. Rev. Lett.* **26** 1604–5
Rosen S P 1971 *J. math. Phys.* **12** 673–81
Sarkar S C 1971 *PhD Thesis* University of London
Weinberg S 1968 *Phys. Rev.* **166** 1568–77